Explicit Computations on the Desingularized Kummer Surface

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Abstract. We find formulas for the birational maps from a Kummer surface $\mathcal{K}$ and its dual $\mathcal{K}^*$ to their common minimal desingularization $\mathcal{S}$. We show how the nodes of $\mathcal{K}$ blow up. Then we give a description of the group of linear automorphisms of $\mathcal{S}$.

Mathematics Subject Classification (2000). Primary 14J28, 14M15; Secondary 14J50.

Keywords. Genus 2 curves; Kummer surfaces; line complexes.

1. Introduction

The Kummer surface is a mathematical object having a long history and which has been considered from various points of view. We present the two approaches of this topic that are relevant for this paper.

In the 19-th century a singular surface $\mathcal{K}$, called the Kummer surface, was attached to a quadratic line complex. A minimal desingularization Σ of $\mathcal{K}$ and a birational map $\kappa_1 : \mathcal{K} \dashrightarrow \Sigma$ were constructed by geometric methods. One may call this the "old", or *classical* construction of the Kummer, which we recall in Section 4.

Another construction is the following: let A be an abelian surface, and let the involution $\sigma : A \longrightarrow A$ be given by $\sigma(x) = -x$. The quotient $\mathcal{K} = A/\sigma$ has 16 double points and one defines a K3 surface $\mathcal{S}$ to be $\mathcal{K}$ with these 16 nodes blown up (see [Be], Prop. 8.11). The presently prevailing terminology in literature designates $\mathcal{S}$ as the Kummer surface attached to A. However in this paper, to be consistent with the historical point of view and with our main reference [CF], we call $\mathcal{S}$ the *desingularized* Kummer surface. If $\mathcal{K} = \mathcal{J}(\mathcal{C})/\sigma$, where $\mathcal{J}(\mathcal{C})$ is the Jacobian of a curve $\mathcal{C}$ of genus 2, then $\mathcal{K}$ is called the Kummer surface *belonging to* $\mathcal{C}$. The natural question of the connection between the two constructions of the Kummer surface was explicitly answered in [CF], Chapter 17: given the equations of $\mathcal{C}$, one can write down the equations of the quadratic

line complex which will yield, by classical construction, the Kummer surface belonging to $\mathcal{C}$ (see Lemma 4.5).

The Jacobian $\mathcal{J}(\mathcal{C})$ can be embedded in $\mathbb{P}^{15}$ and is described by 72 quadratic equations ([Fl]). This makes explicit calculations with the Jacobian difficult, for instance giving equations of twists. Methods have been looked for, to study the Mordell-Weil group of $\mathcal{J}(\mathcal{C})$ using more computable objects.

A desingularization $\mathcal{S}$ of $\mathcal{K}$ is constructed explicitly in [CF], Chapter 16 by algebraic methods. The surface $\mathcal{S}$ appeared naturally in recent attempts to compute the Mordell-Weil group, by using important tools as Cassels' morphism and the fake Selmer group. The K3 surface $\mathcal{S}$ is a smooth intersection of three quadrics in $\mathbb{P}^5$. Denote by $\mathcal{K}^*$ the projective dual of $\mathcal{K}$. There are birational maps

$$\kappa : \mathcal{K} \dashrightarrow \mathcal{S} \quad \text{and} \quad \kappa^* : \mathcal{K}^* \dashrightarrow \mathcal{S}.$$

There are morphisms extending $\kappa^{-1} : \mathcal{S} \dashrightarrow \mathcal{K}$ and ${\kappa^*}^{-1} : \mathcal{S} \dashrightarrow \mathcal{K}^*$ to all of $\mathcal{S}$, which we also denote by κ^{-1} and ${\kappa^*}^{-1}$ and these are minimal desingularizations of $\mathcal{K}$ and $\mathcal{K}^*$.

Origins. Cassels and Flynn explain that the surface $\mathcal{S}$ comes from the behaviour of six of the tropes (see Definition 2.4) under the duplication map. The existence of $\mathcal{S}$ raises more far-reaching questions. Indeed, if the ground field k is algebraically closed, one always has a commutative diagram:

$$\begin{array}{ccc} \mathcal{J}(\mathcal{C}) & \xrightarrow{d_0} & \mathcal{J}(\mathcal{C})^0 \\ \pi \big\downarrow & & \big\downarrow \pi_0 \\ \mathcal{K} & \xrightarrow{d_0^*} & \mathcal{K}^* \end{array} \tag{1.1}$$

where $\mathcal{J}(\mathcal{C})^0$ is the dual of $\mathcal{J}(\mathcal{C})$ as an abelian variety. Here, the maps d_0 and d_0^* depend on the choice of a rational point on $\mathcal{C}$. In other words, the abelian varieties duality matches with the projective one (see [CF], Note 1 on page 35). When k is not algebraically closed, one has to enlarge the ground field to obtain such diagrams, yet $\mathcal{S}$ is a desingularization *over* k of both $\mathcal{K}$ and $\mathcal{K}^*$. One can ask if there is a unifying object for $\mathcal{J}(\mathcal{C})$ and $\mathcal{J}(\mathcal{C})^0$, generalizing the abelian varieties duality.

Recent developments. Perhaps the most natural definition of $\mathcal{S}$ and its twists is related to an idea of M. Stoll and N. Bruin, in connection with the computation of the Mordell-Weil group of $\mathcal{J}(\mathcal{C})$. This was presented a few years after [CF] appeared. We give a brief account of it in Section 5.

Cassels and Flynn already suggested that the 2-Selmer group could be investigated by using twists of $\mathcal{S}$. In 2007 A. Logan and R. van Luijk ([LL]) and P. Corn ([C]) made use of twists of $\mathcal{S}$ to find specific curves with nontrivial 2-torsion elements in the Tate-Shafarevich groups of their Jacobians.

Our results and structure of this paper. In Section 2 we give a background.

In Sections 3 and 4 we achieve part of the program suggested in [CF] at the end of Chapter 16. In Section 3 we complete the construction in [CF], enlarge the set of points where $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ is explicitly defined and find formulae for κ, listed in the Appendix. These formulae allow one to describe how each singularity of $\mathcal{K}$ blows up,

beyond the simple use of the Kummer structure. Our treatment is algebraic, in the vein of the definition of $\mathcal{S}$.

In Section 4 we turn to geometric language. We first recall the classical construction of $\mathcal{K}$. In Proposition 4.10 we give an explicit isomorphism Θ between Σ and $\mathcal{S}$. Then we observe that well-known projective isomorphisms W_i between $\mathcal{K}$ and $\mathcal{K}^*$ lift to Σ to correspondences given by line complexes of degree 1. Using Θ we show that the W_i lift to $\mathcal{S}$ to known commuting involutions (Corollary 4.12). With the formulae for κ at hand one can now find formulae for κ^*.

In Section 5 we give a description of the group of linear automorphisms of $\mathcal{S}$.

This paper is almost self-contained. Moreover, all essential information needed is contained in [CF].

2. Preliminaries

We work over a field k of characteristic different from 2. By a curve, we mean a smooth projective irreducible variety of dimension 1. *Throughout this paper, $\mathcal{C}$ will be a curve of genus* 2. If k has at least 6 elements and $\mathrm{char}(k) \neq 2$, such a curve admits an affine model:

$$\mathcal{C}' : \quad Y^2 = F(X), \tag{2.1}$$

where

$$F(X) = f_0 + f_1 X + \ldots + f_6 X^6 \in k[X], \qquad f_6 \neq 0$$

and F has distinct roots $\theta_1, \ldots, \theta_6$. The points $\mathfrak{a}_i = (\theta_i, 0)$ are the Weierstrass points. We denote by $\infty^{\pm}$ the points at infinity on the completion $\mathcal{C}$ of $\mathcal{C}'$. For a given point $\mathfrak{r} = (x, y)$ on $\mathcal{C}$, the *conjugate of* $\mathfrak{r}$ *under the* $\pm Y$ *involution* is the point $\bar{\mathfrak{r}} = (x, -y)$. In accordance, for a divisor $\mathfrak{X} = \sum n_i \mathfrak{r}_i$ we denote by $\overline{\mathfrak{X}} = \sum n_i \bar{\mathfrak{r}}_i$. The class of a divisor $\mathfrak{X}$ is denoted by $[\mathfrak{X}]$ and a divisor in the canonical class by $K_{\mathcal{C}}$.

There is a bijection between $\mathrm{Pic}^0(\mathcal{C})$ and $\mathrm{Pic}^2(\mathcal{C})$ defined by $[\mathfrak{D}] \mapsto [\mathfrak{D} + K_{\mathcal{C}}]$. Hence one can regard a point of the Jacobian $J(\mathcal{C})$ as the class of a divisor $\mathfrak{X} = \mathfrak{r} + \mathfrak{u}$, where $\mathfrak{r} = (x, y)$, $\mathfrak{u} = (u, v)$ is a pair of points on $\mathcal{C}$.

Effective divisors of degree 2 can be identified with points on the symmetric product $\mathcal{C}^{(2)}$ of $\mathcal{C}$ with itself. This is a non-singular variety, since $\mathcal{C}$ is a non-singular curve. Now, the canonical class is represented by an infinity of divisors $\mathfrak{r} + \bar{\mathfrak{r}}$, while any other class in $\mathrm{Pic}^2(\mathcal{C})$ has a unique representative. Hence the Jacobian should look like $\mathcal{C}^{(2)}$ with the representatives of $[K_{\mathcal{C}}]$ "blown down". Flynn finds a projective emdedding of the Jacobian (see [Fl]) in the following way.

For a point $\mathfrak{X} = \{\mathfrak{r}, \mathfrak{u}\}$ on $\mathcal{C}^{(2)}$ with $\mathfrak{r} = (x, y)$, $\mathfrak{u} = (u, v)$, define:

$$\sigma_0 = 1, \quad \sigma_1 = x + u, \quad \sigma_2 = xu,$$

$$\beta_0 = \frac{F_0(x, u) - 2yv}{(x - u)^2},$$

where

$$\begin{aligned} F_0(x,u) = \quad & 2f_0 + f_1(x+u) + 2f_2xu + f_3xu(x+u) \\ & +2f_4(xu)^2 + f_5(xu)^2(x+u) + 2f_6(xu)^3. \end{aligned}$$

The Jacobian is then the projective locus of $\mathbf{z} = (z_0 : \ldots : z_{15})$ in $\mathbb{P}^{15}$, where $z_0 = \delta$; $z_1 = \gamma_1$; $z_2 = \gamma_0$; $z_i = \beta_{5-i}$, $i = 3,4,5$; $z_i = \alpha_{9-i}$, $i = 6,\ldots,9$; $z_i = \sigma_{14-i}$, $i = 10,\ldots,14$; and $z_{15} = \rho$. For the definition of the functions α, β, etc. and details, see [Fl].

Using the bijection between $\operatorname{Pic}^0(\mathcal{C})$ and $\operatorname{Pic}^2(\mathcal{C})$ one can describe generically the group law $\oplus$ on $\mathcal{J}(\mathcal{C})$, with neutral element $[K_{\mathcal{C}}]$. Let $\mathfrak{U}$, $\mathfrak{B}$ be divisors such that $\mathfrak{U}+\mathfrak{B} \neq \mathfrak{r}+\bar{\mathfrak{r}}+\mathfrak{C}$, with $\mathfrak{r} \in \mathcal{C}(\bar{k})$ and $\mathfrak{C}$ divisor of degree 2. Then there is a unique $M(X) \in \bar{k}[X]$ of degree 3 such that the cubic $Y = M(X)$ passes through the four points of $\mathfrak{U}$, $\mathfrak{B}$. The complete intersection of the cubic curve with $\mathcal{C}$ is given by

$$M(X)^2 = F(X), \qquad Y = M(X).$$

The residual intersection is an effective divisor $\mathfrak{D}$. Then $[\mathfrak{U}] \oplus [\mathfrak{B}] = [\overline{\mathfrak{D}}]$, i.e. $[\mathfrak{U}] \oplus [\mathfrak{B}] = [(x_5, -M(x_5)) + (x_6, -M(x_6))]$, where x_5, x_6 are the last two roots of $M(X)^2 - F(X)$.

Definition 2.1. The Kummer surface $\mathcal{K}$ belonging to a curve of genus 2, is the projective locus in $\mathbb{P}^3$ of the elements $\xi = (\xi_1 : \xi_2 : \xi_3 : \xi_4)$, where

$$\xi_1 = \sigma_0, \quad \xi_2 = \sigma_1, \quad \xi_3 = \sigma_2, \quad \xi_4 = \beta_0. \tag{2.2}$$

The equation of the Kummer surface is given in [CF], formula (3.1.9). It is of the form

$$\mathcal{K}: \quad K = K_2\xi_4^2 + K_1\xi_4 + K_0 = 0 \tag{2.3}$$

where the K_i are forms of degree $4-i$ in ξ_1, ξ_2, ξ_3. The natural map from $J(\mathcal{C})$ to $\mathcal{K}$ given by

$$(z_0 : \ldots : z_{15}) \longmapsto (z_{14} : z_{13} : z_{12} : z_5) = (\xi_1 : \ldots : \xi_4)$$

is 2 to 1; the ramification points correspond to divisor classes $[\mathfrak{X}]$ with $[\mathfrak{X}] = [\overline{\mathfrak{X}}]$. In the sense of the group law of the Jacobian, this means that $2[\mathfrak{X}] = [K_{\mathcal{C}}]$. The images of these classes are the 16 singular points (nodes) on $\mathcal{K}$: $N_0 = (0:0:0:1)$ corresponding to $[K_{\mathcal{C}}]$ and other $\begin{pmatrix} 6 \\ 2 \end{pmatrix} = 15$ nodes N_{ij} corresponding to classes of divisors $\mathfrak{X}_{ij} = \mathfrak{a}_i + \mathfrak{a}_j$ with $i \neq j$.

Definition 2.2. A function $f \in \bar{k}(\mathcal{J}(\mathcal{C}))$ is called *even* if $f([\bar{\mathfrak{r}}]) = f([\mathfrak{r}])$ and *odd* if $f([\bar{\mathfrak{r}}]) = -f([\mathfrak{r}])$.

Definition 2.3. The surface $\mathcal{K}^* \subset (\mathbb{P}^3)^\checkmark = \mathbb{P}^3$ is the projective dual of $\mathcal{K}$, i.e. to a point $\xi \in \mathcal{K}$ corresponds the point $\eta \in \mathcal{K}^*$ such that $\eta = (\eta_1 : \eta_2 : \eta_3 : \eta_4) \in (\mathbb{P}^3)^\checkmark$ gives the tangent plane to $\mathcal{K}$ at ξ.

Definition 2.4. There are 6 planes T_i containing the 6 nodes N_0 and N_{ij}, $j \neq i$ and 10 planes T_{ijk} containing the 6 nodes N_{mn} for $\{m,n\} \subset \{i,j,k\}$ or $\{m,n\} \cap \{i,j,k\} = \emptyset$. These are the *tropes*; they cut conics on $\mathcal{K}$. They correspond to the 16 singular points of $\mathcal{K}^*$.

When using the term trope, it will be clear from the context if we refer to planes, conics or singular points of $\mathcal{K}^*$. The equations of the T_i are given in (3.13).

3. The Desingularized Kummer

We recall the facts from [CF] Chapter 16 we need, keeping the notation there. We start with a simple, yet a bit technical, explanation of the ideas leading to the construction of the desingularized Kummer $\mathcal{S}$. For a more conceptual one, see [CF], Chapter 6, Section 6.

Recall that the Kummer parametrizes divisor of degree 2, modulo linear equivalence and $\pm Y$ involution. Let $[\mathfrak{X}] = [(x,y)+(u,v)] \neq [K_{\mathcal{C}}]$, where $yv \neq 0$ and $x \neq u$ be a divisor class. There is a unique $M(X)$ of degree 3 such that

$$M(X)^2 - F(X) = (X-x)^2(X-u)^2 H(X), \tag{3.1}$$

for a quadratic $H(X)$. The divisor given by $H(X) = 0,\ Y = -M(X)$ is in the class $2[\mathfrak{X}]$. There is a unique polynomial $P^*(X)$ of degree at most 5 such that

$$(X-x)(X-u)P^*(X) \equiv M(X) \bmod F(X). \tag{3.2}$$

Then

$$(X-x)^2(X-u)^2 {P^*}^2(X) \equiv M^2(X) \equiv (X-x)^2(X-u)^2 H(X) \bmod F(X)$$

and since $F(x)F(u) = (yv)^2 \neq 0$, we have also

$${P^*}^2(X) \equiv H(X) \bmod F(X).$$

Changing $\mathfrak{X}$ to $\overline{\mathfrak{X}}$, changes $M(X)$ to $-M(X)$, so $P^*(X)$ to $-P^*(X)$.

Conversely, given P^* with $\deg(P^*) \leq 5$ and $(P^*)^2 \equiv$ quadratic $\bmod F$, the equation

$$(X-x)(X-u)P^*(X) \equiv \text{cubic} = M(X) \bmod F(X) \tag{3.3}$$

puts 2 conditions on x, u, so has in general a unique set of solutions. The divisor classes of $\mathfrak{D} = (x, M(x)) + (u, M(u))$ and $\overline{\mathfrak{D}}$ give the same point on the Kummer and correspond to P^* and $-P^*$. All this suggests the following construction.

Let $\mathbf{p} = (p_0 : \ldots : p_5)$, where the p_j are indeterminates, and put $P(X) = \sum_0^5 p_j X^j$. Let $\mathcal{S}$ the projective locus of the $\mathbf{p}$ for which $P(X)^2$ is congruent to a quadratic in X modulo $F(X)$. Put

$$P_j(X) = \prod_{i \neq j} (X - \theta_i) \tag{3.4}$$

and $\omega_j = P_j(\theta_j) \neq 0$. Since $\theta_i \neq \theta_j$ for $i \neq j$, we have $\omega_j \neq 0$ and the P_j span the vector space of polynomials of degree at most 5. We have

$$P(X) = \sum_j \pi_j P_j(X), \qquad \text{where} \qquad \pi_j = \frac{P(\theta_j)}{\omega_j}. \tag{3.5}$$

The $K3$ surface $\mathcal{S}$ given as the complete intersection in $\mathbb{P}^5$ of the three quadrics $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$ where

$$\mathcal{S}_i : S_i = 0, \qquad \text{and} \qquad S_i = \sum_j \theta_j^i \omega_j \pi_j^2 \quad \text{for} \quad i = 0, 1, 2 \tag{3.6}$$

is a minimal desingularization of $\mathcal{K}$ and also of $\mathcal{K}^*$. Here the S_i are quadratic forms in $\mathbf{p}$ with coefficients in $\mathbb{Z}[f_1, \ldots, f_6]$.

The following theorems hold ([CF], Theorems 16.5.1 and 16.5.3):

Theorem 3.1. *There is a birational map $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ defined for general $\xi \in \mathcal{K}$ as follows:*

Let $\mathfrak{X} = \{(x, y), (u, v)\}$ correspond to ξ. Put $G(X) = (X - x)(X - u)$ and let $M(X)$ be the cubic determined by the property that $Y - M(X)$ vanishes twice on $\mathfrak{X}$. Let $P(X) = \sum_0^5 p_j X^j$ be determined by $GP \equiv M \bmod F$. Then $\kappa(\xi)$ is the point with projective coordinates $(p_0 : \ldots : p_5)$.

Let $\kappa^* : \mathcal{K}^* \dashrightarrow \mathcal{S}$ be the birational map defined in [CF], Theorem 16.5.2.

Theorem 3.2. *Let $\xi \in \mathcal{K}$ and $\eta \in \mathcal{K}^*$ be dual, that is η gives the tangent to $\mathcal{K}$ at ξ. Then $\kappa(\xi) = \kappa^*(\eta)$.*

Our first result is the following.

Lemma 3.3. *The map $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ from Theorem* 3.1 *is given by the formulae listed in the Appendix.*

Proof. The problem is to make effective the method given in [CF], Chapter 16. For completeness and because of typing errors there, we recall it in the Appendix. As presented in [CF], the method works for a general element $[\mathfrak{X}] = [(x, y) + (u, v)]$, where $yv \neq 0$ and $x \neq u$. After finding the polynomial P from Theorem 3.1, one has to modify it slightly, so that its coefficients be even functions of $k(J(\mathcal{C}))$ and therefore belong to the function field of the Kummer.

We get formulae for κ by expressing $y^2 = F(x)$, $v^2 = F(u)$,

$$yv = \frac{F_0(x, u) - \beta_0(x - u)^2}{2}$$

in the coefficients of $P(X)$, then as the resulting coefficients are symmetric fonctions of x and u, we express them in terms of $\xi_2 = x + u$ and $\xi_3 = xu$. Finally we homogenize the formulae with respect to $\xi_1 = 1$, ξ_2, ξ_3, $\xi_4 = \beta_0$.

One first obtains

$$\kappa(\xi) = (\tilde{p}_0(\xi) : \ldots : \tilde{p}_5(\xi)),$$

where

$$\tilde{p}_j(\xi) = \alpha_j K_2 + \beta_j (K_1 \xi_4 + K_0), \quad \text{for } 0 \leq j \leq 5. \tag{3.7}$$

Here α_j and β_j are homogeneous forms in ξ of degree 4 and 2 respectively and the K_j are those in (2.3).

Taking $p_j(\xi) = (\tilde{p}_j(\xi) - \beta_j K)/K_2 = \alpha_j - \beta_j \xi_4^2$, we obtain formulae of degree 4 for κ which will be defined also for $K_2 = 0$, extending κ to images of divisor classes $[\mathfrak{X}] = [(x, y) + (u, v)]$ with $x = u$ and $y = v \neq 0$. However the formulae do not work for points with $F'(x) = 0$ and for the image of $[2\infty^+]$. We will treat the case $y = 0$ or $v = 0$ in connection with nodes and tropes. $\square$

There are 6 commuting involutions $\varepsilon^{(i)}$ of $\mathcal{S}$, which can be described as follows. Let the point $(p_0 : \ldots : p_5)$ be represented by the polynomial $P(X) = p_0 + p_1 X + \cdots p_5 X^5$ and let

$$g_i(X) = 1 - 2\frac{P_i(X)}{P_i(\theta_i)}, \tag{3.8}$$

where $P_i(X)$ is defined by (3.4). We see that $g_i(\theta_j) = (-1)^{\delta_{ij}}$, so $g_i(X)^2 \equiv 1 \mod F(X)$. Then one defines

$$\varepsilon^{(i)}(P(X)) = g_i(X)P(X) \mod F(X). \tag{3.9}$$

In terms of coordinates π_j, one has

$$\varepsilon^{(i)}(\pi_j) = (-1)^{\delta_{ij}}\pi_j. \tag{3.10}$$

Definition 3.4. We define $\mathrm{Inv}(\mathcal{S})$ to be the group of 32 commuting involutions of S generated by the $\varepsilon^{(i)}$.

The $\mathbf{p} = (p_0 : p_1 : 0 : 0 : 0 : 0)$ are clearly in $\mathcal{S}$ and form a rational line Δ_0. We shall often write $p_0 + p_1 X \in \Delta_0$. Acting on Δ_0 by the involutions gives 31 further lines.

Notation 3.5. We denote:

$$\Delta_i = \varepsilon^{(i)}(\Delta_0)\,, \quad \Delta_{ij} = \varepsilon^{(i)} \circ \varepsilon^{(j)}(\Delta_0) \quad \text{and} \quad \Delta_{ijk} = \varepsilon^{(i)} \circ \varepsilon^{(j)} \circ \varepsilon^{(k)}(\Delta_0)\,.$$

We come now to the main result in this section, which describes how the singularities of $\mathcal{K}$ and $\mathcal{K}^*$ blow up.

Lemma 3.6. *The map κ blows up the node $N_0 = (0 : 0 : 0 : 1)$ of $\mathcal{K}$ into the line Δ_0 and the* 15 *nodes N_{ij} into the lines Δ_{ij}. The tropes T_i and T_{ijk} blow up by κ^* into the lines Δ_i and Δ_{ijk}.*

Note. This result is predicted in [CF] as plausible.

Proof. The node N_0 corresponds to the canonical class, therefore we consider divisors of the type $\mathfrak{X} = (x, y) + (u, v)$ with $u = x + h$, h small and $v \approx -y \neq 0$. Then the local behaviour of the Kummer coordinates is $\xi_1 = 1$, $\xi_2 = 2x + h \approx 2x$, $\xi_3 = x(x + h) \approx x^2$ and

$$\xi_4 = \frac{F_0(x, x+h) - 2yv}{h^2} \approx \frac{4y^2}{h^2}.$$

Replacing this in the formulae for κ and clearing denominators, then taking the limit as $h \to 0$, we obtain

$$\begin{aligned} \kappa(\xi) &\approx (-16xy^4 : 16y^4 : 0 : 0 : 0 : 0) \\ &\approx (-x : 1 : 0 : 0 : 0 : 0), \end{aligned}$$

since $y \neq 0$.

Note that for $(X - \theta_i) \in \Delta_0$, we have

$$\varepsilon^{(i)}(X - \theta_i) \equiv g_i(X)(X - \theta_i) \equiv (X - \theta_i) \mod F(X),$$

so Δ_0 and Δ_i intersect at $(-\theta_i : 1 : 0 : 0 : 0 : 0)$.

We now show that $\Delta_0 \cap \Delta_{ij} = \emptyset$ for $i \neq j$. Indeed, the intersection point $\mathfrak{p}$ should be invariant by $\varepsilon^{(i)} \circ \varepsilon^{(j)}$. A polynomial $P(X)$ represents such a point iff

$$\alpha P(X) \equiv g_i(X)g_j(X)P(X) \mod F(X) \quad \text{for some } \alpha \in \bar{k}^*$$
$$\text{iff}$$
$$F(X) \mid P(X)(\alpha - g_i(X)g_j(X)).$$

Replacing X by the roots of $F(X)$ one sees that $P(X)$ must have at least two roots among the θ_k, so it must be of degree at least 2 and therefore cannot represent a point on Δ_0. Similarly, $\Delta_0 \cap \Delta_{ijk} = \emptyset$ for $i \neq j \neq k$.

The six Δ_i are strict transforms of the conics cut on $\mathcal{K}$ by the tropes containing N_0. To see this, recall that we still have to define κ for points corresponding to divisors $\mathfrak{X} = \{x, y\} + \{\theta_i, 0\}$ with $y \neq 0$. Write $F(X) = f_6(X - \theta_i)P_i(X)$. From this we get formulae for f_k, $k = 0, \ldots, 6$ depending on θ_i and h_{ij}, $j = 0, \ldots 5$, the coefficients of $P_i(X)$, which we plug into

$$\xi_4 = \frac{F_0(x, \theta_i)}{(x - \theta_i)^2}.$$

We substitute then $\xi_1 = 1$, $\xi_2 = x + \theta_i$, $\xi_3 = x\theta_i$ and ξ_4 in the formulae for κ. On multiplying by $(x - \theta_i)^2/(f_6^2 P_i(x))$ (note that $P_i(x) \neq 0$), we obtain

$$P(X) = 2(x - \theta_i)P_i(X) + P_i(\theta_i)(X - x), \tag{3.11}$$

that is

$$\begin{array}{lcll} p_0 & = & 2h_{i0}(x - \theta_i) - P_i(\theta_i)x & \\ p_1 & = & 2h_{i1}(x - \theta_i) + P_i(\theta_i) & \\ p_j & = & 2h_{ij}(x - \theta_i) & \text{for } 2 \leq j \leq 5. \end{array} \tag{3.12}$$

The points $(1 : x + \theta_i : x\theta_i : \xi_4)$ belong to the conic T_i cut on $\mathcal{K}$ by the trope

$$\theta_i^2 \xi_1 - \theta_i \xi_2 + \xi_3 = 0, \tag{3.13}$$

passing throught N_0 and N_{ij}, $j \neq i$. Formulae (3.12) give parametric equations (in x) of the strict transform by κ of this conic. To confirm that this is Δ_i, one verifies that

$$P(X) \equiv P_i(\theta_i)g_i(X)(X - x) \mod F(X).$$

Recall from [CF], Chapter 4, Section 5 that there are linear maps $W_i : \mathcal{K} \longrightarrow \mathcal{K}^*$, taking the node N_0 to the trope T_i (W_i is induced by addition of a Weierstrass point $\mathfrak{a}_i$). Further, $W_i^{-1} \circ W_j$ moves N_0 to the node N_{ij}. Applying the results in Section 4 and especially Corollary 4.12, one concludes that:

1) the tropes T_i considered as singular points of $\mathcal{K}^*$, blow up by κ^* into Δ_i;
2) each of the fifteen N_{ij} blows up into Δ_{ij};
3) the tropes T_{ijk}, $i \neq j \neq k$ blow up into Δ_{ijk};
4) the ten Δ_{ijk} ($i \neq j \neq k$) are strict transforms of the ten conics cut on $\mathcal{K}$ by the tropes (planes) not containing N_0. Each of them intersects six Δ_{ij} since each node is on six tropes. □

4. Linear and quadratic line complexes

We recall from [Hu] and [CF] the definitions of the Kummer surface and of the corresponding desingularization Σ in terms of quadratic complexes. Then we link them to the surface S studied in Section 3. Notations are like in [CF], Chapter 17.

If $u = (u_1 : u_2 : u_3 : u_4)$ and $v = (v_1 : v_2 : v_3 : v_4)$ are distinct points in $\mathbb{P}^3$, then the Grassman coordinates of the line $< u, v > \subset \mathbb{P}^3$ are

$$\mathfrak{p} = (p_{43} : p_{24} : p_{41} : p_{21} : p_{31} : p_{32}) = (X_1 : \ldots : X_6),$$

with $p_{ij} = u_i v_j - u_j v_i$. Denote by $\mathcal{G}$ the Grassmanian quadric in $\mathbb{P}^5$, representing the lines in $\mathbb{P}^3$. Its equation is

$$G(X_1, \dots, X_6) = 2X_1X_4 + 2X_2X_5 + 2X_3X_6 = 0.$$

Definition 4.1. A line complex of degree d is a set of lines in $\mathbb{P}^3$ whose Grassman coordinates satisfy a homogeneous equation $Q(X_1, \dots, X_6) = 0$ of degree d.

If $d = 1$ this is called a linear complex and if $d = 2$ a quadratic complex.

A line $L \in \mathcal{G}$ parametrizes a pencil of lines in $\mathbb{P}^3$. The lines of a pencil L all pass through a point $\mathfrak{f}(L) = u$, called the *focus* of the pencil) and lie in one plane $\mathfrak{h}(L) = \pi_u$, the *plane* of the pencil.

All lines in a linear complex $\mathcal{L}$ passing through a given point u (respectively lying in a plane π), form a pencil L_u (respectively L_π). Each linear complex $\mathcal{L}$ establishes a *correspondence* between points and planes in $\mathbb{P}^3$:

$$I(u) = \mathfrak{h}(L_u), \quad I(\pi) = \mathfrak{f}(L_\pi), \quad I^2 = 1,$$

which is defined also for lines; if $l \subset \mathbb{P}^3$ is the line $< u, u' >$, then $I(l) = I(u) \cap I(u')$. The line $I(l)$ is the *polar* line of l with respect to the given linear complex.

Definition 4.2. Two linear complexes are called apolar if the correspondences they define commute.

Let H be any quadratic form in six variables such that the quadrics $G = 0$ and $H = 0$ intersect transversely and denote by $\mathcal{H} = \{x \in \mathbb{P}^5 \mid H(x) = 0\}$. Let $\mathcal{W} = \mathcal{G} \cap \mathcal{H}$ and $\mathcal{A} =$ set of lines on $\mathcal{W}$. The points in $\mathcal{W}$ represent the lines in $\mathbb{P}^3$ whose Grassman coordinates $\mathfrak{p}$ satisfy $H(\mathfrak{p}) = 0$. A line $L \in \mathcal{A}$ represents a pencil of lines of this quadratic complex in $\mathbb{P}^3$.

Definition 4.3. The Kummer surface $\mathcal{K} \subset \mathbb{P}^3$ associated to the quadratic complex $\mathcal{H}$ is the locus of focuses of such pencils: $\mathcal{K} = \{\mathfrak{f}(L) \mid L \in \mathcal{A}\}$.

Definition 4.4. The dual Kummer surface $\mathcal{K}^* \subset \mathbb{P}^{3\checkmark}$ associated to the quadratic complex $\mathcal{H}$ is the locus of planes of such pencils.

From now on we suppose $f_6 = 1$.

Lemma 4.5. *For any curve $\mathcal{C}$ of genus 2, the Kummer surface belonging to the curve $\mathcal{C}$ given by (2.1) coincides with the Kummer surface just defined, if one takes the quadratic complex $\mathcal{H}$ to be given by*

$$\begin{aligned} H = \quad & -4X_1X_5 - 4X_2X_6 - X_3^2 + 2f_5X_3X_6 + 4f_0X_4^2 \\ & +4f_1X_4X_5 + 4f_2X_5^2 + 4f_3X_5X_6 + (4f_4 - f_5^2)X_6^2. \end{aligned}$$

Proof. See [CF], Lemma 17.3.1 and pages $182 - 183$. □

Now, if a point $\xi \in \mathbb{P}^3$ is the focus of the pencil corresponding to the line $L_\xi \in \mathcal{A}$, then L_ξ lies in the plane $\Pi_\xi \subset \mathcal{G}$ corresponding to lines in $\mathbb{P}^3$ passing through ξ. But then the conic $\Pi_\xi \cap \mathcal{H}$ contains L_ξ, so is degenerate; Π_ξ is tangent to $\mathcal{H}$ and $\Pi_\xi \cap \mathcal{H} = L_\xi \cup L'_\xi$. The lines of the quadratic complex passing through ξ are in the two pencils L_ξ and L'_ξ,

each with focus ξ, lying in the planes π_ξ and π'_ξ in $\mathbb{P}^3$. The point ξ is a *singular point* of the quadratic complex. The line $l_\xi = \pi_\xi \cap \pi'_\xi$ is represented on $\mathcal{G}$ by the point $\mathfrak{p}_\xi = L_\xi \cap L'_\xi$ and is called a *singular line* of the quadratic complex.

If $L_\xi \neq L'_\xi$ the pencils are distinct and ξ is a simple point of the Kummer; there is a one-to-one correspondence $\xi \leftrightarrow \mathfrak{p}_\xi$. However, if $L_\xi = L'_\xi$, then $\pi_\xi = \pi'_\xi$ and all the lines in L_ξ are singular lines. The point ξ is a singular point of the Kummer, because the map $\mathfrak{f} : \mathcal{A} \longrightarrow \mathcal{K}$ is algebraic. Therefore the variety Σ parametrizing singular lines is a desingularization of the Kummer.

Remark 4.6. The quadrics $\mathcal{S}_0$ and $\mathcal{S}_2$ in the defining equation (3.6) of $\mathcal{S}$ are dual with respect to $\mathcal{S}_1$ and Cassels and Flynn call for an interpretation of this duality. More precisely, if $Q \in \mathcal{S}_0$ (respectively $Q \in \mathcal{S}_2$) then the hyperplane

$$\sum_{j=0}^{5} \frac{\partial S_1}{\partial \pi_j}(Q) \cdot \pi_j = 0$$

is tangent to $\mathcal{S}_2$, respectively to $\mathcal{S}_0$. Now, it is shown in [Hu], Section 31 that, if one brings G and H to diagonal form :

$$G : \sum_{i=1}^{6} {X_i}^2 = 0 \qquad \text{and} \qquad H : \sum_{i=1}^{6} \alpha_i {X_i}^2 = 0,$$

then the variety parametrizing the singular lines is

$$\Sigma = \mathcal{G} \cap \mathcal{H} \cap \mathcal{F} \qquad \text{where} \qquad \mathcal{F} : \sum_{i=1}^{6} \alpha_i^2 {X_i}^2 = 0$$

(see also [CF], Corollary 2 to Lemma 17.2.1). Over $\bar{k}$ one can change $\pi_j \leftrightarrow \sqrt{\omega_j}\pi_j$ and so the equations of $\mathcal{S}_0$ and $\mathcal{S}_1$ give that of $\mathcal{S}_2$. This explains the duality of $\mathcal{S}_0$ and $\mathcal{S}_2$ with respect to $\mathcal{S}_1$.

Definition 4.7. The birational map $\kappa_1 : \mathcal{K} \dashrightarrow \Sigma$ is defined by $\kappa_1(\xi) = \mathfrak{p}_\xi$.

Definition 4.8. The birational map $\kappa_1^* : \mathcal{K}^* \dashrightarrow \Sigma$ associates to a plane π tangent to $\mathcal{K}$ the intersection point of the lines in $\mathcal{A}$ parametrizing the two pencils in $\mathcal{H}$ contained in π.

It is shown in [CF] and [Hu] that κ_1^{-1} and ${\kappa_1^*}^{-1}$ extend to minimal desingularizations $\kappa_1^{-1} : \Sigma \longrightarrow \mathcal{K}$ and ${\kappa_1^*}^{-1} : \Sigma \longrightarrow \mathcal{K}^*$.

Lemma 4.9. *The surface $\mathcal{K}^*$ is the projective dual of $\mathcal{K}$ that is, if $\xi = \mathfrak{f}(L) \in \mathcal{K}$ then $\eta = \mathfrak{h}(L) \in \mathcal{K}^*$ is the tangent plane of $\mathcal{K}$ at ξ. Therefore $\kappa_1(\xi) = \kappa_1^*(\eta)$.*

Proof. See [CF], page 181. □

4.1. Connection between $\mathcal{S}$ and Σ

Denote by $G(\overrightarrow{X}, \overrightarrow{Y})$ the bilinear form associated to the Grassmanian G. Make the change of coordinates

$$\zeta_i = \frac{G(\overrightarrow{X}, \overrightarrow{v}(\theta_i))}{\sqrt{\omega_i}}, \tag{4.1}$$

with vectors $\overrightarrow{v}(\theta_i)$ as in [CF] formula (17.4.3). The desingularisation of the Kummer surface corresponding to the quadratic complex H of Lemma 4.5 is the $K3$ surface Σ given as the complete intersection in $\mathbb{P}^5$ of the three quadrics Σ_0, Σ_1, Σ_2 where

$$\Sigma_i : \qquad \sum_j \theta_j^i \zeta_j^2 = 0 \quad \text{for} \quad i = 0, 1, 2.$$

(see also [Hu], Section 31).

Let $\Theta : \Sigma \longrightarrow \mathcal{S}$ be defined by

$$\Theta(\zeta_1 : \cdots : \zeta_6) = \left(\frac{\zeta_1}{\sqrt{\omega_1}} : \cdots : \frac{\zeta_6}{\sqrt{\omega_6}} \right) = (\pi_1 : \cdots : \pi_6). \tag{4.2}$$

To write Θ in variables X_j on Σ and p_j on $\mathcal{S}$, recall that $P(X) = \sum_0^5 p_j X^j$ and note that by (3.5) and (4.1):

$$\frac{P(\theta_i)}{\omega_i} = \pi_i = \frac{\zeta_i}{\sqrt{\omega_i}} = \frac{G(\overrightarrow{X}, \overrightarrow{v}(\theta_i))}{\omega_i}.$$

Now, as polynomials in X, we have $G(\overrightarrow{X}, \overrightarrow{v}(X)) = P(X)$, because they have degree 5 and agree on the six θ_i. Explicit formulae for Θ are

$$\begin{array}{lll} p_0 = X_1 + f_1 X_4 & p_2 = X_3 + 2f_4 X_5 + 2f_3 X_4 + f_5 X_6 & p_4 = 2f_5 X_4 + 2X_5 \\ p_1 = X_2 + 2f_2 X_4 + f_3 X_5 & p_3 = 2f_4 X_4 + 2f_5 X_5 + 2X_6 & p_5 = 2X_4. \end{array}$$

Proposition 4.10. *Denoting by κ^{-1} and κ_1^{-1} the blow-downs from $\mathcal{S}$, respectively Σ to $\mathcal{K}$ one has $\kappa_1^{-1} = \kappa^{-1} \circ \Theta$.*

Proof. Pick a point $\xi \in \mathbb{P}^3$ and write the equations of the plane $\Pi_\xi \subset \mathcal{G}$ of lines through ξ (see (4.7)). Take $\mathcal{H}$ to be defined as in Lemma 4.5. As seen, Π_ξ is tangent to $\mathcal{H}$ iff the intersection consists of two lines:

$$\Pi_\xi \cap \mathcal{H} = L_\xi \cup L'_\xi.$$

Computing in terms of ξ the coordinates of $\mathfrak{p}_\xi = L_\xi \cap L'_\xi$, we find homogeneous formulae for X_i in ξ_i of degree 4:

$$\mathfrak{p}_\xi = (X_1(\xi) : \cdots : X_6(\xi)) = \kappa_1(\xi).$$

We compare now

$$\Theta \circ \kappa_1(\xi) = (\hat{p}_0(\xi) : \cdots : \hat{p}_5(\xi)) : \mathcal{K} \dashrightarrow \mathcal{S}$$

with $\kappa(\xi) = (p_0(\xi) : \cdots : p_5(\xi))$ from Lemma 3.3 and obtain

$$\hat{p}_i p_5 - \hat{p}_5 p_i = \delta_i K \qquad \text{with } K \text{ given by (2.3)},$$

for δ_i a homogeneous polynomial in ξ. □

Associated with a quadratic complex $\mathcal{H} : H = 0$ there is a set of 6 mutually apolar linear complexes $\mathcal{L}_k$, such that the polar of any line in $\mathcal{H}$ with respect to $\mathcal{L}_k$ is in $\mathcal{H}$. If G and H are written in diagonal form, these complexes are

$$\mathcal{L}_k : \zeta_k = 0 \quad \text{for} \quad k = 1, \ldots 6.$$

The action of the correspondences I_k on lines in $\mathbb{P}^3$ translates in coordinates $\zeta = (\zeta_1 : \ldots : \zeta_6)$ by

$$I_k(\zeta_i) = (-1)^{\delta_{ik}}\zeta_i, \tag{4.3}$$

which restricts to Σ. The Kummer is determined by $\mathcal{H}$, so must be invariant under the transformation I_k. Therefore the set of nodes and tropes is invariant (see [Hu], Section 30).

If $u \in \mathbb{P}^3$, we have $I_k(u) = \mathfrak{h}(L_{k,u})$ in our previous notation. Let $I_{jk} = I_j \circ I_k$. Since

$$I_j \circ I_{jk}(u) = I_k(u),$$

we have $I_k(u) =$ plane of lines in $\mathcal{L}_j$ passing through $I_{jk}(u)$. So, if N is a node, each plane $I_k(N)$ passes both through the nodes N and $I_{jk}(N)$ for $j \neq k$, so is a trope.

Now let W_i be as in the proof of Lemma 3.6.

Proposition 4.11. *For any k, the map I_k is the unique automophism of Σ such that the following diagram is commutative:*

$$\begin{tikzcd} \Sigma \arrow[r, "I_k"] \arrow[d, "\kappa_1^{-1}"'] & \Sigma \arrow[d, "\kappa_1^{*-1}"] \\ \mathcal{K} \arrow[r, "W_k"'] & \mathcal{K}^*. \end{tikzcd} \tag{4.4}$$

Proof. Let $\xi \in \mathcal{K}$ be a simple point and denote $\mathfrak{p}_\xi = \kappa_1(\xi)$. For a subset $V \subset \mathcal{G}$, put

$$I_k(V) = \{I_k(l) \in \mathcal{G} \mid l \in V\}.$$

The pencils $I_k(L_\xi)$ and $I_k(L'_\xi)$ are both contained in the polar plane of ξ with respect to $\mathcal{L}_k$, which by Lemma 4.14 is $W_k(\xi)$. The plane in $\mathbb{P}^5$ parametrizing lines in $W_k(\xi)$ is therefore tangent to $\mathcal{H}$ at $I_k(L_\xi) \cap I_k(L'_\xi) = I_k(L_\xi \cap L'_\xi) = I_k(\mathfrak{p}_\xi) = I_k \circ \kappa_1(\xi)$. By definition of κ_1^* we have $\kappa_1^*(W_k(\xi)) = I_k \circ \kappa_1(\xi)$. □

The following corollary illustrates how the projective duality (over $k(\theta_k)$) between $\mathcal{K}$ and $\mathcal{K}^*$ lifts to $\mathcal{S}$.

Corollary 4.12. *For any k, the map $\varepsilon^{(k)}$ is the unique automophism of $\mathcal{S}$ such that the following diagram is commutative:*

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\varepsilon^{(k)}"] \arrow[d, "\kappa^{-1}"'] & \mathcal{S} \arrow[d, "\kappa^{*-1}"] \\ \mathcal{K} \arrow[r, "W_k"'] & \mathcal{K}^*. \end{tikzcd} \tag{4.5}$$

Proof. Let $\xi \in \mathcal{K}$ and $\eta \in \mathcal{K}^*$ be dual. We have :

$$\Theta \circ \kappa_1^*(\eta) \overset{4.9}{=} \Theta \circ \kappa_1(\xi) \overset{4.10}{=} \kappa(\xi) \overset{3.2}{=} \kappa^*(\eta). \tag{4.6}$$

Note that $\Theta \circ I_k \circ \Theta^{-1} = \varepsilon^{(k)}$, by (4.2), (4.3) and (3.10). Therefore

$$\kappa^{*-1} \circ \varepsilon^{(k)} \overset{(4.6)}{=} \kappa_1^{*-1} \circ \Theta^{-1} \circ \Theta \circ I_k \circ \Theta^{-1} \overset{4.11}{=} W_k \circ \kappa_1^{-1} \circ \Theta^{-1} \overset{4.10}{=} W_k \circ \kappa^{-1}.$$

This is summarized in the following diagram

$$\begin{array}{ccccccc}
\mathcal{S} & \xleftarrow{\Theta} & \Sigma & \xrightarrow{I_k} & \Sigma & \xrightarrow{\Theta} & \mathcal{S} \\
& \searrow & \downarrow & & \downarrow & \swarrow & \\
& & \mathcal{K} & \underset{W_k}{\longrightarrow} & K^* & &
\end{array}$$

□

Now Corollary 4.12 is useful for finding explicit formulae for κ^*, simply because

$$\kappa^* = \kappa^* \circ W_i \circ \kappa^{-1} \circ \kappa \circ W_i^{-1} = \varepsilon^{(i)} \circ \kappa \circ W_i^{-1}$$

on an open dense set in $\mathcal{K}^*$. The resulting formulae are huge and not listed in this paper, since on a given example it is much easier to apply successively each map involved.

We now look for the formulae for W_i. Since we need only one transformation W_i to find κ^*, we may suppose that *all* the roots of F are non-zero, for else we use the formulae given in [CF], Chapter 4.

Lemma 4.13. *Let θ_i be a root of F and recall that $f_6 = 1$. Then the transformation $W_i : \mathcal{K} \longrightarrow \mathcal{K}^*$ corresponding to the addition of the Weierstrass point $(\theta_i, 0)$ has the following antisymmetric matrix:*

$$A_i = \begin{pmatrix} 0 & -f_1 - 2\frac{f_0}{\theta_i} & a_{13} & \theta_i^2 \\ f_1 + 2\frac{f_0}{\theta_i} & 0 & \theta_i^2(f_5 + 2\theta_i) & -\theta_i \\ -a_{13} & -\theta_i^2(f_5 + 2\theta_i) & 0 & 1 \\ -\theta_i^2 & \theta_i & -1 & 0 \end{pmatrix}$$

where $a_{13} = \theta_i(f_3 + 2f_4\theta_i + 2f_5\theta_i^2 + 2\theta_i^3)$.

Note. Each time the vector of coordinates of a point in $\mathbb{P}^3$ is involved in matrix or scalar product computations, we view it as a column vector.

Proof. Suppose we want the matrix corresponding to W_1. Recall from [CF], Chapter 3 what the nodes N_{ij} are :

$$N_{ij} = (1 : \theta_i + \theta_j : \theta_i\theta_j : \beta_0(i,j)),$$

where

$$\beta_0(i,j) = -\prod_{m \neq i,j} \theta_m - \theta_i\theta_j(\theta_i\theta_j + \sum_{s \neq t} \theta_s\theta_t)$$

and in the last sum $s, t \notin \{i, j\}$.

On writing that $W_1(N_0) = T_1 = (\theta_1^2 : -\theta_1 : 1 : 0)$ one finds: $a_{14} = \theta_1^2$, $a_{24} = -\theta_1$, $a_{34} = 1$ and $a_{44} = 0$. Now looking at the last coordinate of the equality $W_1(N_{1i}) = T_i$ one gets a relation:

$$a_{41} + a_{42}(\theta_1 + \theta_i) + a_{43}\theta_1\theta_i + a_{44}\beta_0(1,i) = 0,$$

with a simple solution $a_{41} = -\theta_1^2$, $a_{42} = \theta_1$, $a_{43} = -1$ and $a_{44} = 0$. One considers then the equality $W_1(N_{ij}) = T_{1ij}$, where the trope T_{1ij} has coordinates

$$T_{1ij} = \begin{pmatrix} (\theta_1 + \theta_i + \theta_j)\theta_k\theta_l\theta_m + \theta_1\theta_i\theta_j(\theta_k + \theta_l + \theta_m) : & -\theta_1\theta_i\theta_j - \theta_k\theta_l\theta_m : \\ \theta_1\theta_i + \theta_1\theta_j + \theta_i\theta_j + \theta_k\theta_l + \theta_k\theta_m + \theta_l\theta_m : & 1 \end{pmatrix},$$

where k, l, m are the indices from 1 to 6 different from $1, i, j$. This yields

$$a_{41} + a_{42}(\theta_i + \theta_j) + a_{43}\theta_i\theta_j + a_{44}\beta_0(i,j) = 1$$

projectively. The value of the last expression is

$$v = -\theta_1^2 + \theta_1(\theta_i + \theta_j) - \theta_i\theta_j$$

and this corresponds projectively to 1, so the coordinates we want to find are those of the trope T_{1ij}, each multiplied by v. One finishes the computations using Viète's formulae and antisymmetry. □

Lemma 4.14. *For any point $\xi \in \mathbb{P}^3$ the plane with dual coordinates $W_i(\xi)$ is the polar plane of ξ with respect to $\mathcal{L}_i$.*

Proof. To make a choice, put $i = 1$. Put $W_1(\xi) = w = (w_1 : \ldots : w_4)$. We have

$$\sum w_i\xi_i = \xi^T w = \xi^T A_1 \xi = 0 \quad \text{since } A_1 \text{ is antisymmetric,}$$

so the plane with dual coordinates w passes through ξ.

Now, a line with Grassmann coordinates $(X_1 : \ldots : X_6)$ passes through a point $\xi = (\xi_1 : \ldots : \xi_4) \in \mathbb{P}^3$ iff the following relations hold :

$$\begin{cases} \xi_1 X_6 - \xi_2 X_5 + \xi_3 X_4 = 0 \\ \xi_1 X_2 + \xi_2 X_3 - \xi_4 X_4 = 0 \\ \xi_1 X_1 - \xi_3 X_3 + \xi_4 X_5 = 0. \end{cases} \tag{4.7}$$

Similarly, such a line lies in the plane

$$\Pi : \quad \sum_{i=1}^{4} a_i \xi_i = 0$$

in $\mathbb{P}^3$ iff

$$\begin{cases} a_2 X_4 + a_3 X_5 + a_4 X_3 = 0 \\ a_1 X_4 - a_3 X_6 + a_4 X_2 = 0 \\ a_1 X_5 + a_2 X_6 - a_4 X_1 = 0. \end{cases} \tag{4.8}$$

Note that the entries in the matrix A_1 corresponding to W_1 are :

$$a_{12} = v_1, \quad a_{13} = v_2, \quad a_{14} = v_6, \quad a_{23} = v_3, \quad a_{34} = v_4, \quad a_{42} = v_5,$$

where the v_i are those defined in [CF], formula (17.4.3) and $\theta = \theta_1$. The dual coordinates of the plane $W_1(\xi)$ are

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = A_1 \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} v_1\xi_2 + v_2\xi_3 + v_6\xi_4 \\ -v_1\xi_1 + v_3\xi_3 - v_5\xi_4 \\ -v_2\xi_1 - v_3\xi_2 + v_4\xi_4 \\ -v_6\xi_1 + v_5\xi_2 - v_4\xi_3 \end{pmatrix}.$$

Now, considering relation (4.8) with $a_i = w_i$, the conditions that a line passing through ξ with Grassmann coordinates $(X_1 : \ldots : X_6)$ lie in the plane $W_1(\xi)$ all reduce to :

$$v_1 X_4 + v_4 X_1 + v_2 X_5 + v_5 X_2 + v_3 X_6 + v_6 X_3 = 0$$

(one has to take into account also (4.7)). But, up to a constant factor this is the value of ζ_1 under the change of variables which brings G and H from Lemma 4.5 to diagonal form (cf. (4.1) or [CF], formula (17.4.5)). Therefore, a line that passes through ξ is contained in $W_1(\xi)$ iff it belongs to $\mathcal{L}_1$ defined by $\zeta_1 = 0$. This is exactly what we want. $\square$

5. Twist of the desingularized Kummer

We denote by $\mathfrak{S} = \mathcal{J}(k)$ the Mordell-Weil group of the Jacobian. There is a map defined by Cassels ([Cas]):

$$\Phi : \mathfrak{S} \longrightarrow \mathcal{L} = L^*/k^*(L^*)^2 \quad \text{where} \quad L = k[T]/(F(T)).$$

The fake Selmer group is defined by Poonen and Schaefer in [PS]:

$$\mathrm{Sel}^{(2)}_{\mathrm{fake}}(k, \mathcal{J}) = \{\xi \in \mathcal{L} \mid \mathrm{res}_\nu(\xi) \in \Phi_\nu(\mathcal{J}(k_\nu)),\ \text{for all } \nu \in \Omega\},$$

where Ω is the set of places of k, $\mathcal{L}_\nu = L_\nu^*/k_\nu^*(L_\nu^*)^2$, $L_\nu = L \otimes_k k_\nu$ and $\Phi_\nu : \mathcal{J}(k_\nu) \longrightarrow \mathcal{L}_\nu$. The restriction map $\mathrm{res}_\nu : \mathcal{L} \longrightarrow \mathcal{L}_\nu$ is induced by $k \hookrightarrow k_\nu$.

We present now an idea of M. Stoll and N. Bruin. Let $\xi \in \mathrm{Sel}^{(2)}_{\mathrm{fake}}(k, \mathcal{J})$; one looks for an element

$$[D] = \{(x, y), (u, v)\} \in \mathfrak{S}$$

such that $\Phi([D]) = \xi$. Denote by $\xi(X)$ a representative polynomial of degree 5 of ξ.

The formula $\Phi([D]) = \xi$ can now be written as:

$$n\,(x - X)(u - X) \equiv \xi(X)\,P(X)^2 \mod F(X),$$

where $P(X)$ is the polynomial of degree 5 to be found and $n \in k^*$.

Define $\mathcal{S}^\xi \subset \mathbb{P}^5$ as the projective locus of polynomials $P(X)$ of degree 5 such that $\xi(X)P(X)^2 \equiv$ quadratic $\mod F(X)$. The surface $\mathcal{S}^\xi$ is given by the three quadratic forms in the coefficients of $P(X)$:

$$S^\xi \;:\qquad C_5^\xi = C_4^\xi = C_3^\xi = 0,$$

where

$$C^\xi(X) = C_5^\xi X^5 + C_4^\xi X^4 + C_3^\xi X^3 + C_2^\xi X^2 + C_1^\xi X + C_0^\xi$$

is the polynomial such that

$$C^\xi(X) \equiv \xi(X)\,P(X)^2 \mod F(X). \tag{5.1}$$

To a rational point on $\mathcal{S}^\xi$ corresponds a quadratic rational polynomial which is a candidate for being of the form $\Phi([D])$ for $[D] \in \mathfrak{S}$.

If we take $\xi = 1$, we obtain the desingularized Kummer $\mathcal{S}$.

One can interpret this construction as giving a twist of $\mathcal{S}$ in the following way. If $\beta(X) \in \bar{k}[X]$ is a polynomial of degree 5 such that

$$\beta(X)^2 \equiv \xi(X) \mod F(X),$$

the twist is given by the isomorphism $\alpha : \mathcal{S}^\xi \longrightarrow \mathcal{S}$, where

$$P(X) \longmapsto \alpha(P(X)) \equiv \beta(X)P(X) \mod F(X).$$

The twist $\mathcal{S}^\xi$ can be diagonalized like $\mathcal{S}$. We keep the notations $P_j(X)$, ω_j and π_j from Section 3. Putting

$$\xi_j = \xi(\theta_j),$$

we also have

$$\xi(X) = \sum_{j=1}^{6} \frac{\xi_j}{\omega_j} P_j(X).$$

Taking into account that

$$\begin{aligned} P_j(X)^2 &\equiv \omega_j P_j(X) \mod F(X), \\ P_i(X)\, P_j(X) &\equiv 0 \mod F(X) \quad \text{for } i \neq j, \end{aligned} \tag{5.2}$$

finally gives

$$\xi(X)\, P(X)^2 \equiv \sum_{j=1}^{6} \xi_j \omega_j \pi_j^2 P_j(X) \mod F(X).$$

Since

$$\begin{aligned} P_j(X) &= F(X)/\,(f_6(X-\theta_j) \\ &= X^5 + (\theta_j + (f_5/f_6))X^4 + (\theta_j^2 + (f_5/f_6)\theta_j + (f_4/f_6))X^3 + \cdots, \end{aligned}$$

the surface S^ξ is obtained in the variables π_j as the intersection of the three quadrics $S_i^\xi = 0$ ($i = 0, 1, 2$), where

$$\begin{array}{lclcl} S_0^\xi & = & C_5^\xi & = & \sum_j \xi_j \omega_j \pi_j^2, \\ S_1^\xi & = & f_6 C_4^\xi - f_5 C_5^\xi & = & f_6 \sum_j \theta_j \xi_j \omega_j \pi_j^2, \\ S_2^\xi & = & f_6^2 C_3^\xi - f_5 f_6 C_4^\xi + (f_5^2 - f_4 f_6) C_5^\xi & = & f_6^2 \sum_j \theta_j^2 \xi_j \omega_j \pi_j^2. \end{array}$$

Note that if $\xi = 1$ then $\xi_j = 1$ for all j.

6. Linear automorphisms of $\mathcal{S}$

Keeping Notation 3.5, we let

$$\begin{aligned} \mathfrak{p}_i &= \Delta_0 \cap \Delta_i, \\ \mathfrak{p}_{ij} &= \Delta_i \cap \Delta_{ij} = \varepsilon^{(i)}(\mathfrak{p}_j), \\ \mathfrak{p}_{ijk} &= \Delta_{ij} \cap \Delta_{ijk} = \varepsilon^{(i)}(\mathfrak{p}_{jk}). \end{aligned} \tag{6.1}$$

Remark 6.1. Since there are no other lines on $\mathcal{S}$ (see [GH], page 775), this is the whole structure of line intersections on $\mathcal{S}$.

Let $\mathrm{GL}(\mathcal{S})$ be the group of linear automorphisms of $\mathcal{S}$.

Lemma 6.2. *Let $A, B \in \mathrm{GL}(\mathcal{S})$ such that $A_{|\Delta_0} = B_{|\Delta_0}$. Then $A = B$.*

Proof. Let $I \in \mathrm{GL}(\mathcal{S})$ be the identity. If $A \in \mathrm{GL}(\mathcal{S})$ and $A_{|\Delta_0} = I_{|\Delta_0}$, then A fixes the $\mathfrak{p}_i$, so invaries the Δ_i. But then A invaries also Δ_{ij}, the unique line other than Δ_0 which meets Δ_i and Δ_j, so A fixes $\mathfrak{p}_{ij}$, $j = 1, \ldots 6$. Hence $A_{|\Delta_i} = I_{|\Delta_i}$. Similarly, one sees that A is the identity on any of the 32 lines on $\mathcal{S}$, so $A = I$. $\square$

Let $A \in \mathrm{GL}(\mathcal{S})$. Since $A(\Delta_0)$ is a line, by Remark 6.1 there exists a unique involution $\varepsilon \in \mathrm{Inv}(\mathcal{S})$ such that $\varepsilon \circ A(\Delta_0) = \Delta_0$. We associate to A the permutation $\sigma \in S_6$ such that

$$\varepsilon \circ A(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)} \quad \text{for } i = 1, \ldots 6. \tag{6.2}$$

Note that $\sigma = \mathrm{id}$ iff $\varepsilon \circ A_{|\Delta_0} = I_{|\Delta_0}$ iff $\varepsilon \circ A = I$ (by Lemma 6.2) iff $A \in \mathrm{Inv}(\mathcal{S})$.

Definition 6.3. $\mathrm{GL}_0(\mathcal{S})$ is the subgroup of $\mathrm{GL}(\mathcal{S})$ of linear automorphisms A such that $A(\Delta_0) = \Delta_0$.

Lemma 6.4. *Let $A \in \mathrm{GL}(\mathcal{S})$ and $\sigma \in S_6$ be the permutation associated to A by (6.2). Then, for any $1 \leq i \leq 6$ we have:*

$$A \circ \varepsilon^{(i)} = \varepsilon^{(\sigma(i))} \circ A. \tag{6.3}$$

Proof. Let $B = \varepsilon \circ A$. Then $B(\Delta_0) = \Delta_0$ and $B(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)}$, so $B(\Delta_i) = \Delta_{\sigma(i)}$. The unique line cutting $\Delta_{\sigma(i)}$ and $\Delta_{\sigma(j)}$ is $\Delta_{\sigma(i)\sigma(j)}$ hence, $B(\Delta_{ij}) = \Delta_{\sigma(i)\sigma(j)}$. Then

$$B(\mathfrak{p}_{ij}) = B(\Delta_i \cap \Delta_{ij}) = B(\Delta_i) \cap B(\Delta_{ij}) = \Delta_{\sigma(i)} \cap \Delta_{\sigma(i)\sigma(j)} = \mathfrak{p}_{\sigma(i)\sigma(j)}.$$

Now one sees that $(\varepsilon \circ A)^{-1} \circ \varepsilon^{(\sigma(i))} \circ (\varepsilon \circ A)$ acts like $\varepsilon^{(i)}$ on $\mathfrak{p}_j$. By Lemma 6.2 and knowing that $\mathrm{Inv}(\mathcal{S})$ is commutative, we conclude $A \circ \varepsilon^{(i)} = \varepsilon^{(\sigma(i))} \circ A$. □

Proposition 6.5. *Let $\psi : \mathrm{GL}(\mathcal{S}) \longrightarrow \mathrm{GL}_0(\mathcal{S})$ be the map $A \mapsto \varepsilon \circ A$ defined by formula (6.2). We have an exact sequence of groups*

$$1 \longrightarrow \mathrm{Inv}(\mathcal{S}) \longrightarrow \mathrm{GL}(\mathcal{S}) \xrightarrow{\psi} \mathrm{GL}_0(\mathcal{S}) \longrightarrow 1.$$

Proposition 6.5 implies that $\mathrm{Inv}(\mathcal{S})$ is a normal subgroup of $\mathrm{GL}(\mathcal{S})$, being the kernel of ψ.

Corollary 6.6. *For any linear automorphism A of $\mathcal{S}$ not in $\mathrm{Inv}(\mathcal{S})$, the centralizer of A in $\mathrm{Inv}(\mathcal{S})$ is not equal to $\mathrm{Inv}(\mathcal{S})$.*

We now show that $\mathrm{GL}_0(\mathcal{S})$ is in bijection with the group of linear automorphisms of Δ_0 which invary the set $\{p_i,\, i = 1, \ldots 6\}$.

Proposition 6.7. *Let $\sigma \in S_6$ and $B : \Delta_0 \longrightarrow \Delta_0$ a linear automorphism of Δ_0 such that for $1 \leq i \leq 6$, we have $B(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)}$. Then there exists a unique $A \in \mathrm{GL}_0(\mathcal{S})$ such that $A_{|\Delta_0} = B$.*

Proof. Suppose σ and B given. If A exists, it is unique by Lemma 6.2 and σ is the permutation associated to A defined by (6.2). Let $\widetilde{A}$ the linear operator of $\mathcal{P}_5$ (polynomials of degree ≤ 5) associated to A. Let $a, b, c, d \in \bar{k}$ such that

$$\widetilde{A}(1) = aX + b \quad \text{and} \quad \widetilde{A}(X) = cX + d\,.$$

After some linear algebra and using (6.3), we find that the image of a point $\mathfrak{p} \in \mathcal{S}$ represented by

$$P(X) = \sum_j \pi_j P_j(X)\,,$$

is

$$\widetilde{A}(P(X)) = \sum_j \underbrace{\left(\pi_j \frac{\omega_j}{\omega_{\sigma(j)}}(a\theta_{\sigma(j)} + b)\right)}_{\pi'_{\sigma(j)}} P_{\sigma(j)}(X) \,. \tag{6.4}$$

We have to prove that the point $(\pi'_1 : \ldots : \pi'_6)$ satisfies the equations (3.6).

We show that $k_{\sigma(j)}\omega_{\sigma(j)}{\pi'}^2_{\sigma(j)} = \alpha_j\omega_j\pi_j^2$ for a quadratic polynomial α_j in θ_j. We have:

$$k_{\sigma(j)}\omega_{\sigma(j)}{\pi'}^2_{\sigma(j)} = k_{\sigma(j)}(a\theta_{\sigma(j)} + b)^2 \frac{\omega_j}{\omega_{\sigma(j)}} \omega_j \pi_j^2 \,,$$

and then

$$\alpha_j = k_{\sigma(j)}(a\theta_{\sigma(j)} + b)^2 \frac{\omega_j}{\omega_{\sigma(j)}} \,.$$

One can write $\widetilde{A}(X - \theta_i)$ in two ways, using the fact that $A(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)}$ or linearity of $\widetilde{A}$:

$$\mu_j(X - \theta_{\sigma(j)}) = \widetilde{A}(X - \theta_j) = cX + d - \theta_j(aX + b) \quad \text{where } \mu_j \in \bar{k} \,.$$

Replacing $X = \theta_{\sigma(j)}$, we obtain the formula

$$\theta_j = \frac{c\theta_{\sigma(j)} + d}{a\theta_{\sigma(j)} + b} \,, \tag{6.5}$$

which gives the relations between the roots of $F(X)$ necessary for the existence of the linear automorphism B.

Now, we calculate ω_j replacing each θ_j by the formula (6.5):

$$\begin{aligned}
\omega_j &= \prod_{i \neq j}(\theta_i - \theta_j) = \prod_{i \neq j}\left(\frac{c\theta_{\sigma(i)} + d}{a\theta_{\sigma(i)} + b} - \frac{c\theta_{\sigma(j)} + d}{a\theta_{\sigma(j)} + b}\right) \\
&= \frac{1}{(a\theta_{\sigma(j)} + b)^4} \underbrace{\frac{1}{\prod_i (a\theta_{\sigma(i)} + b)}}_{\text{constant}} \prod_{i \neq j}\left((\theta_{\sigma(i)} - \theta_{\sigma(j)}) \underbrace{(bc - ad)}_{\text{constant}}\right) .
\end{aligned}$$

Call γ the constant part of the equation:

$$\frac{\omega_j}{\omega_{\sigma(j)}} = \frac{\gamma}{\left(a\theta_{\sigma(j)} + b\right)^4} \,. \tag{6.6}$$

Replacing (6.6) in α_j, we have:

$$\alpha_j = k_{\sigma(j)}(a\theta_{\sigma(j)} + b)^2 \frac{\gamma}{\left(a\theta_{\sigma(j)} + b\right)^4} = \gamma \frac{k_{\sigma(j)}}{\left(a\theta_{\sigma(j)} + b\right)^2} \,.$$

To see that α_j is a quadratic polynomial in θ_j (for each k_j), we use the formula (6.5) to obtain:

$$a\theta_j - c = \frac{ad-bc}{a\theta_{\sigma(j)}+b}$$

which gives the result for $k_{\sigma(j)} = 1$;

$$a^2\theta_j^2 - c^2 = \frac{2ac(ad-bc)\theta_{\sigma(j)} + a^2d^2 - b^2c^2}{(a\theta_{\sigma(j)}+b)^2}$$

which gives the result for $k_{\sigma(j)} = \theta_{\sigma(j)}$;

$$b\theta_j - d = \frac{(bc-ad)\theta_{\sigma(j)}}{a\theta_{\sigma(j)}+b}$$

which gives the result for $k_{\sigma(j)} = \theta_{\sigma(j)}^2$.

□

Proposition 6.7 gives necessary and sufficient conditions for the existence of non-trivial elements of $\mathrm{GL}_0(\mathcal{S})$. For the case of non-commuting involutions of $\mathcal{S}$, we can write this conditions easily.

Remark 6.8. For a curve of genus 2 defined by

$$Y^2 = F(X) = \prod_{i=1}^{6}(X-\theta_i),$$

we may suppose (up to a linear translation) that

$$\theta_3 + \theta_4 = \theta_5 + \theta_6 \quad \text{or} \quad \theta_3\theta_4 = \theta_5\theta_6.$$

Indeed, suppose that $\theta_3 + \theta_4 \neq \theta_5 + \theta_6$. On putting $\tilde{\theta}_i = \theta_i + t$ with

$$t = \frac{\theta_5\theta_6 - \theta_3\theta_4}{(\theta_3+\theta_4)-(\theta_5+\theta_6)}.$$

one gets

$$\tilde{\theta}_3\tilde{\theta}_4 = \tilde{\theta}_5\tilde{\theta}_6 \,.$$

Note that if $\theta_3+\theta_4 = \theta_5+\theta_6$ and $\theta_3\theta_4 = \theta_5\theta_6$ then $\{\theta_3, \theta_4\} = \{\theta_5, \theta_6\}$, which is impossible.

Corollary 6.9. *Let A be a non-commuting involution of $\mathcal{S}$ which fixes Δ_0. Renumbering the roots of F, we have*

$$A(\mathfrak{p}_j) = \mathfrak{p}_{j+1}$$

for $j = 3, 5$. By Remark 6.8, we may suppose that $\theta_3+\theta_4 = \theta_5+\theta_6$ or $\theta_3\theta_4 = \theta_5\theta_6$. Then:

If $A(\mathfrak{p}_1) = \mathfrak{p}_2$, we have

$$\theta_1 + \theta_2 = \theta_3 + \theta_4 = \theta_5 + \theta_6 \quad \textit{or} \quad \theta_1\theta_2 = \theta_3\theta_4 = \theta_5\theta_6.$$

Otherwise, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are fixed by A and then $\theta_1 + \theta_2 = 0$ and

$$\theta_1^2 = \theta_2^2 = \theta_3\theta_4 = \theta_5\theta_6.$$

Appendix

Construction of κ

As announced during the proof of Lemma 3.3, we give the needed polynomials for the construction of κ. Let $[\mathfrak{X}] = [(x,y)+(u,v)] \neq [K_{\mathcal{C}}]$, where $yv \neq 0$ and $x \neq u$ be a divisor class.

There is a unique $M(X)$ of degree 3 such that

$$M(X)^2 - F(X) = (X-x)^2(X-u)^2 H(X), \tag{6.7}$$

for a quadratic $H(X)$. There is a unique polynomial $P(X)$ of degree at most 5 such that

$$(X-x)(X-u)P(X) \equiv M(X) \bmod F(X). \tag{6.8}$$

Let

$$\begin{aligned} M(X) &= (m_x(X-x)+1)\left((X-u)/(x-u)\right)^2 y \\ &\quad + (m_u(X-u)+1)\left((X-x)/(u-x)\right)^2 v, \end{aligned} \tag{6.9}$$

where m_x amd m_u are given by the conditions that the derivative of $F(X) - M(X)^2$ vanishes at $X = x$ and $X = u$. Hence

$$m_x = \frac{F'(x)}{2F(x)} - \frac{2}{x-u}, \qquad m_u = \frac{F'(u)}{2F(u)} - \frac{2}{u-x}. \tag{6.10}$$

Then consider

$$\begin{aligned} M^{\diamond}(X) &= 2(x-u)^3 yv M(X) \\ &= (X-u)^2 v\left((F'(x)(x-u) - 4F(x))(X-x) + 2F(x)(x-u)\right) \\ &\quad -(X-x)^2 y\left((F'(u)(u-x) - 4F(u))(X-u) + 2F(u)(u-x)\right). \end{aligned} \tag{6.11}$$

All the terms on the r.h.s are divisible by $(X-x)(X-u)$, except $F(x)(x-u)(X-u)^2$ and $F(u)(u-x)(X-x)^2$. But $F(X)-F(x) = (X-x)F(x,X)$ for some polynomial $F(x,X)$ and similarly for $F(X)-F(u)$. Replacing $F(x)$ and $F(u)$ in (6.11), one finds that

$$M^{\diamond}(X) \equiv (X-x)(X-u)P^{\diamond}(X) \bmod F(X),$$

where

$$\begin{aligned} P^{\diamond}(X) &= (F'(x)(x-u) - 4F(x) - 2F(x,X)(x-u))(X-u)v \\ &\quad - (F'(u)(u-x) - 4F(u) - 2F(u,X)(u-x))(X-x)y. \end{aligned}$$

The coefficient of X^6 in $P^{\diamond}$ is $-2f_6(x-u)(y+v)$, so the polynomial

$$P^{\triangle}(X) = P^{\diamond}(X) + 2f_6(x-u)(y+v)F(X)$$

is of degree 5. Both terms on the r.h.s. are odd and antisymmetric in (x,y), (u,v), so multiplying by

$$\frac{y-v}{(x-u)^2},$$

we get a polynomial $P(X)$ whose coefficients are even symmetric functions of (x,y), (u,v), so are even functions of $k(J(\mathcal{C}))$.

We have

$$\begin{aligned} P^2(X) &\equiv \left(P^{\triangle}(X)\frac{y-v}{(x-u)^2}\right)^2 \equiv \left(P^{\diamond}(X)\frac{y-v}{(x-u)^2}\right)^2 \\ &\equiv 4(x-u)^2(y-v)^2y^2v^2H(X) \mod F(X), \end{aligned}$$

where $H(X)$ is as in (6.7). In particular, the polynomial $P(X)$ satisfying the relation (6.8) is the polynomial $P(X) = P^{\triangle}(X)/\bigl(2(x-u)^3yv\bigr)$. One proceeds now as in the proof of Lemma 3.3.

Polynomial definition of κ

$$\begin{aligned} p_0 = \;& -f_3f_6\xi_1\xi_3^3 + 1/2f_5^2\xi_2\xi_3^3 - 2\xi_3^3f_4f_6\xi_2 + 2\xi_3^2\xi_1^2f_1f_6 - \xi_3^2\xi_1^2f_5f_2 - \\ & 2\xi_3^2\xi_1\xi_2f_6f_2 - 1/2\xi_3^2\xi_1f_5\xi_4 - 1/2\xi_3^2\xi_1\xi_2f_5f_3 - \xi_3^2\xi_2^2f_6f_3 - 2\xi_3^2\xi_2f_6\xi_4 \\ & -1/2\xi_3f_3\xi_4\xi_1^2 - 3/2\xi_3\xi_1^2\xi_2f_5f_1 - \xi_3\xi_2f_4\xi_4\xi_1 - 3\xi_3\xi_1\xi_2^2f_6f_1 \\ & -1/2\xi_3\xi_2^2f_5\xi_4 + f_1f_2\xi_1^4 + \xi_1^3\xi_2f_1f_3 + 3/2\xi_1^3f_1\xi_4 + \xi_2^2f_4f_1\xi_1^2 + \xi_2^3f_5f_1\xi_1 \\ & +\xi_2^4f_1f_6 - 1/2\xi_1\xi_2\xi_4^2 \end{aligned}$$

$$\begin{aligned} p_1 = \;& 2\xi_1^4f_2^2 - 2\xi_3\xi_1\xi_2^2f_6f_2 + 1/2\xi_1^2\xi_2^2f_5f_1 - 1/2\xi_1^4f_3f_1 + 2\xi_2^4f_2f_6 + 3\xi_1^3\xi_4f_2 \\ & +1/2\xi_3f_3^2\xi_1^3 + 1/2\xi_2^3f_5\xi_4 + \xi_3\xi_1^2\xi_2f_4f_3 - 1/2\xi_3^2\xi_2^2f_5^2 + 3/2\xi_3\xi_1\xi_2^2f_5f_3 \\ & +2\xi_3^2f_4f_6\xi_2^2 - \xi_3\xi_1^2\xi_2f_5f_2 + \xi_3\xi_2^2f_6\xi_4 + 2\xi_3\xi_2^3f_6f_3 + \xi_1^2\xi_4^2 + 2\xi_1^3\xi_2f_2f_3 \\ & -\xi_3\xi_1^2\xi_2f_6f_1 + 2\xi_1^2f_2f_4\xi_2^2 + 3/2\xi_1^2\xi_2f_3\xi_4 + \xi_1f_4\xi_4\xi_2^2 + \xi_1\xi_2^3f_6f_1 \\ & +2\xi_1\xi_2^3f_5f_2 + 2\xi_3^2\xi_1^2f_6f_2 - 1/2\xi_3^2\xi_1^2f_5f_3 + \xi_3^2\xi_4f_6\xi_1 \end{aligned}$$

$$\begin{aligned} p_2 = \;& 2\xi_1^2\xi_2^2f_4f_3 - f_6f_5\xi_1\xi_3^3 + \xi_3^2\xi_1^2f_3f_6 - \xi_3^2\xi_1^2f_4f_5 + \xi_3^2\xi_1f_5^2\xi_2 \\ & +\xi_3f_1f_6\xi_1^3 + \xi_3\xi_1^3f_3f_4 - 2\xi_3f_5f_2\xi_1^3 + 2\xi_3\xi_1^2\xi_2f_4^2 - 2\xi_3f_5\xi_4\xi_1^2 \\ & -\xi_1^4f_1f_4 + 2\xi_1^4f_3f_2 - 2\xi_3\xi_2f_6f_2\xi_1^2 - 5\xi_3\xi_2^2f_6f_3\xi_1 + 2\xi_3\xi_1\xi_2^2f_5f_4 \\ & -3\xi_3f_6\xi_4\xi_2\xi_1 - 3\xi_3\xi_2f_5f_3\xi_1^2 + 2\xi_2^4f_3f_6 + \xi_2^3f_6\xi_4 + 2f_3\xi_4\xi_1^3 + 2\xi_2f_3^2\xi_1^3 \\ & +2\xi_2^3f_5f_3\xi_1 - \xi_2f_5f_1\xi_1^3 - \xi_2^2f_6f_1\xi_1^2 + \xi_2^2f_5\xi_4\xi_1 + \xi_2f_4\xi_4\xi_1^2 + 2\xi_3\xi_2^3f_6f_4 \\ & +\xi_3^2\xi_2^2f_6f_5 - 4\xi_3^2\xi_1f_4f_6\xi_2 \end{aligned}$$

$$\begin{aligned} p_3 = \;& -2f_6^2\xi_1\xi_3^3 - \xi_3^2\xi_1^2f_5^2 + 2\xi_3^2\xi_1^2f_4f_6 - \xi_3^2\xi_2f_6f_5\xi_1 + 2\xi_3^2f_6^2\xi_2^2 \\ & +\xi_3\xi_1^3f_5f_3 - 2\xi_3\xi_1^3f_2f_6 - \xi_3\xi_1^2\xi_2f_6f_3 - 2\xi_3\xi_1^2f_6\xi_4 + 2\xi_3\xi_1\xi_2^2f_5^2 \\ & -4\xi_3\xi_1f_4f_6\xi_2^2 + 2\xi_3f_6f_5\xi_2^3 - \xi_1^4f_1f_5 + 2\xi_1^4f_4f_2 + 2\xi_1^3f_4\xi_4 + 2\xi_1^3\xi_2f_4f_3 \\ & -\xi_1^3\xi_2f_6f_1 + \xi_1^2\xi_2f_5\xi_4 + 2\xi_1^2f_4^2\xi_2^2 + \xi_1\xi_2^2f_6\xi_4 + 2\xi_1\xi_2^3f_5f_4 + 2\xi_2^4f_6f_4 \end{aligned}$$

$$\begin{aligned} p_4 = \;& \xi_3^2\xi_1^2 f_6 f_5 - 2\xi_3^2\xi_2 f_6^2\xi_1 + \xi_3 f_3 f_6 \xi_1^3 - 2\xi_3\xi_2 f_5^2\xi_1^2 + 2\xi_3 f_4 f_6 \xi_2 \xi_1^2 \\ & -2\xi_3\xi_2^2 f_6 f_5 \xi_1 + 2\xi_3 f_6^2 \xi_2^3 - f_6 f_1 \xi_1^4 + 2 f_5 f_2 \xi_1^4 + 2 f_5 \xi_4 \xi_1^3 + 2\xi_2 f_5 f_3 \xi_1^3 \\ & +2\xi_2^2 f_5 f_4 \xi_1^2 + \xi_2 f_6 \xi_4 \xi_1^2 + 2\xi_2^3 f_5^2 \xi_1 + 2\xi_2^4 f_6 f_5 \end{aligned}$$

$$\begin{aligned} p_5 = \;& 2(f_6\xi_1^2\xi_3^2 - \xi_3 f_5 \xi_2 \xi_1^2 - 2\xi_3 f_6 \xi_2^2 \xi_1 + f_2 \xi_1^4 + \xi_1^3 \xi_4 + \xi_1^3 f_3 \xi_2 + f_4 \xi_2^2 \xi_1^2 \\ & +\xi_2^3 f_5 \xi_1 + f_6 \xi_2^4) f_6 \end{aligned}$$

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